Frobenius Green functors

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Topological Motivation: Morava K-theory and finite groups

For each prime p and each natural number n there is a 2-periodic multiplicative cohomology theory $K_n^*(-)$ with coefficient ring

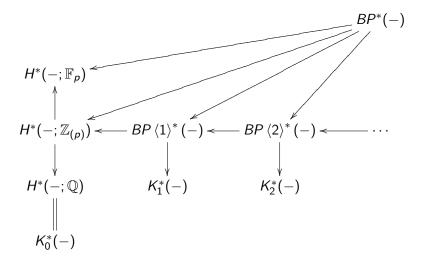
$$\mathcal{K}_n^*(\text{point}) = \mathcal{K}_n^* = \mathbb{F}[u, u^{-1}],$$

where $u \in K_n^{-2}$ and \mathbb{F} is either \mathbb{F}_{p^n} or $\overline{\mathbb{F}}_p$. Here K_n^* is a graded field in an obvious sense; computations are often simplified with the aid of a strict Künneth formula for products:

$$\mathcal{K}_n^*(X \times Y) \cong \mathcal{K}_n^*(X) \otimes_{\mathcal{K}_n^*} \mathcal{K}_n^*(Y).$$

For each finite group G, $K_n^*(BG)$ is finite dimensional over K_n^* and $K_n^*(BG)$ is self-dual as a $K_n^*(BG)$ -module and so is self-injective. Since $K_n^*(BG)$ is also a local K_n^* -algebra, this means that $K_n^*(BG)$ is a local Frobenius algebra. Another important observation is that $\operatorname{Tr}_1^G(1) \neq 0$, and it follows that $\operatorname{Tr}_1^G(1)$ is a basis element for the socle of $K_n^*(BG)$. $K_n^*(B-)$ is a *Green functor* on subgroups of a fixed finite group G.

p-primary field-based cohomology theories

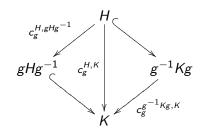


Mackey functors and Green functors for finite groups

Let *G* be a finite group. Consider the category \mathbf{SubGp}_G whose objects are the subgroups of *G* and the morphism set $\mathbf{SubGp}_G(H, K)$ consists of all conjugation maps

$$c_g = c_g^{H,K} \colon H \longrightarrow K; \quad x \mapsto g x g^{-1}$$

for $g \in G$ where $gHg^{-1} \leqslant K$. Here $c_g^{H,K}$ can be factorized as



where $c_g^{H,gHg^{-1}}$ and $c_g^{g^{-1}Kg,K}$ are isomorphisms and \hookrightarrow denotes the inclusion of a subgroup.

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Let \Bbbk be a commutative ring. Roughly speaking, a *Mackey functor* M for G consists of a contravariant functor M^* and a covariant functor M_* on **SubGp**_G taking values in (left) \Bbbk -modules, where

$$\boldsymbol{M}(H) = \boldsymbol{M}^*(H) = \boldsymbol{M}_*(H),$$

and there are various axioms describing the interaction between these. This amounts to the following. For $K \leq H \leq G$ and $g \in G$ there are morphisms

$$\mathsf{res}_{K}^{H}: \boldsymbol{M}(H) \longrightarrow \boldsymbol{M}(K), \qquad \mathsf{ind}_{K}^{H}: \boldsymbol{M}(K) \longrightarrow \boldsymbol{M}(H), \\ c_{g}: \boldsymbol{M}(H) \longrightarrow \boldsymbol{M}(gHg^{-1}).$$

These satisfy the following conditions.

(**MF1**) For $H \leq G$ and $h \in H$, $\operatorname{res}_{\mu}^{H} = \operatorname{ind}_{\mu}^{H} = c_{h} = \operatorname{id}: \boldsymbol{M}(H) \longrightarrow \boldsymbol{M}(H).$ (**MF2**) For $L \leq K \leq H \leq G$. $\operatorname{res}_{L}^{K} \operatorname{res}_{K}^{H} = \operatorname{res}_{L}^{H}, \quad \operatorname{ind}_{K}^{H} \operatorname{ind}_{L}^{K} = \operatorname{ind}_{L}^{H}.$ (**MF3**) For $g_1, g_2 \in G$ and $H \leq G$. $c_{g_1}c_{g_2} = c_{g_1g_2} \colon \boldsymbol{M}(H) \longrightarrow \boldsymbol{M}(g_1g_2Hg_2^{-1}g_1^{-1}).$ (**MF4**) For $K \leq H \leq G$ and $g \in G$.

 $\operatorname{res}_{gKg^{-1}}^{gHg^{-1}}c_g = c_g\operatorname{res}_K^H, \quad \operatorname{ind}_{gKg^{-1}}^{gHg^{-1}}c_g = c_g\operatorname{ind}_K^H.$

(MF5) (Mackey double coset/decomposition formula) For $H \leq G$ and $K \leq H \geq L$,

$$\operatorname{res}_{L}^{H}\operatorname{ind}_{K}^{H} = \sum_{g : L \setminus G / K} \operatorname{ind}_{L \cap g K g^{-1}}^{L} c_{g} \operatorname{res}_{g^{-1} L g \cap K}^{K},$$

where the sum is over a complete set of representatives for the set of double cosets $L \setminus G/K$.

Such a Mackey functor A is a *Green functor* if furthermore A(H) is a \Bbbk -algebra for $H \leq G$, and the following conditions are satisfied. (**GF1**) For $K \leq H \leq G$ and $g \in G$, res^{*H*}_{*K*} and c_g are \Bbbk -algebra homomorphisms.

(**GF2**) (*Frobenius axiom*) For $K \leq H \leq G$, $x \in A(K)$ and $y \in A(H)$,

$$\operatorname{ind}_{K}^{H}(x\operatorname{res}_{K}^{H}(y)) = \operatorname{ind}_{K}^{H}(x)y, \quad \operatorname{ind}_{K}^{H}(\operatorname{res}_{K}^{H}(y)x) = y\operatorname{ind}_{K}^{H}(x).$$

Notice that A(K) becomes both a left and a right A(H)-module via res^{*H*}_{*K*}(*y*), so for $x \in A(K)$ and $y \in A(H)$,

$$y \cdot x = \operatorname{res}_{K}^{H}(y)x, \quad x \cdot y = x \operatorname{res}_{K}^{H}(y).$$

The Frobenius axiom asserts that $\operatorname{ind}_{K}^{H}: A(K) \longrightarrow A(H)$ is both a left and a right A(H)-module homomorphism.

It is often useful to consider extensions of a Mackey functor or Green functor to a larger category.

We say that K is a section of a group G if there is a subgroup $H \leq G$ and an epimorphism $H \longrightarrow K$.

Suppose that \mathcal{X},\mathcal{Y} are two collections of finite groups where \mathcal{X} satisfies the following conditions.

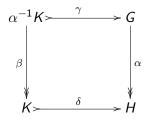
• If $G \in \mathcal{X}$ and if K is a section of G, then $K \in \mathcal{X}$.

• Let
$$G', G'' \in \mathcal{X}$$
. If

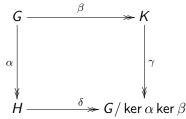
$$1 \rightarrow G' \longrightarrow G \longrightarrow G'' \rightarrow 1,$$

is a short exact sequence, then $G \in \mathcal{X}$.

A globally defined Mackey functor with respect to \mathcal{X}, \mathcal{Y} on finite groups and taking values in \mathbb{K} -modules, is an assignment of an \mathbb{K} -module M(G) to each finite group G, and for each homomorphism $\alpha : G \longrightarrow H$ with ker $\alpha \in \mathcal{X}$ a homomorphism $\alpha^* : M(H) \longrightarrow M(G)$, and for each homomorphism $\beta : K \longrightarrow L$ with ker $\alpha \in \mathcal{Y}$ a homomorphism $\beta_* : M(K) \longrightarrow M(L)$, satisfying the following conditions. (**GD1**) When these are defined, $(\alpha_1 \alpha_2)^* = \alpha_2^* \alpha_1^*$ and $(\beta_1 \beta_2)_* = (\beta_1)_* (\beta_2)_*$. (**GD2**) If $\gamma: G \longrightarrow G$ is an inner automorphism, then $\alpha^* = id = \alpha_*$. (**GD3**) For a commutative diagram of finite groups



we have $\delta^* \alpha_* = \beta_* \gamma^*$ whenever ker $\alpha \in \mathcal{Y}$, and $\alpha^* \delta_* = \gamma_* \beta^*$ whenever ker $\alpha \in \mathcal{X}$. (**GD4**) For a commutative diagram of epimorphisms of finite groups



with ker $\alpha \in \mathcal{Y}$ and ker $\beta \in \mathcal{X}$, we have $\alpha_*\beta^* = \delta^*\gamma_*$. (**GD5**) (Mackey formula) For subgroups $H \leq G \geq K$ with inclusion maps $\iota_H^G \colon H \longrightarrow G$, etc, and conjugation maps $c_g = g(-)g^{-1} \colon g^{-1}Kg \cap H \longrightarrow K \cap gHg^{-1}$,

$$(\iota_{K}^{G})^{*}(\iota_{H}^{G})_{*} = \sum_{g: K \setminus G/H} (\iota_{K \cap gHg^{-1}}^{K})_{*} c_{g}(\iota_{g^{-1}Kg \cap H}^{H})^{*}.$$

Such a globally defined Mackey functor **A** is a *globally defined Green functor* if

(**GD6**) Whenever $\alpha: G \longrightarrow H$ is a homomorphism for which α^* is defined, then $\alpha^*: A(H) \longrightarrow A(G)$ is an k-algebra homomorphism. (**GD7**) (Frobenius axiom) Suppose that $\beta: K \longrightarrow L$ is a homomorphism for which β^* and β_* are both defined. Note that β^* induces left and right A(H)-module structures on A(G) (these coincide when A(G) is commutative). Then $\beta_*: A(G) \longrightarrow A(H)$ is a homomorphism of left and right A(H)-modules.

For our purposes we will take \mathcal{X} to consist of all finite groups, and \mathcal{Y} to consist of either trivial groups or all groups of order not divisible by a positive prime $p = \operatorname{char} \mathbb{k}$.

Now we assume that A is a globally defined Green functor taking values in the category of local Artinian \Bbbk -algebras, for \Bbbk a field with char $\Bbbk = p > 0$. The unique maximal left ideal, $\mathfrak{m}(G) \triangleleft A(G)$, agrees with the Jacobson radical, $\mathfrak{m}(G) = \operatorname{rad} A(G)$, which is a two-sided ideal.

We take \mathcal{X} to consist of all finite groups and \mathcal{Y} to consist of the trivial groups. The restriction/inflation homomorphism associated to a homomorphism $\alpha \colon G \longrightarrow H$ will be denoted

 $\alpha^* : \mathbf{A}(H) \longrightarrow \mathbf{A}(G)$ and the induction homomorphism associated to a monomorphism $\beta : K \longrightarrow L$ by $\beta_* : \mathbf{A}(K) \longrightarrow \mathbf{A}(L)$. When β is the inclusion of a subgroup we will also write $\beta^* = \operatorname{res}_K^L$ and $\beta_* = \operatorname{ind}_K^L$.

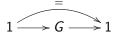
Here α^* is local algebra homomorphism, *i.e.*, $\alpha^*\mathfrak{m}(H) \subseteq \mathfrak{m}(G)$, while β_* is a left or right A(L)-module homomorphism with respect to the A(L)-module structure induced on A(K) by β^* .

We make some further assumptions.

Assumption (\mathbf{A})

For any trivial group 1, A(1) = k.

Notice that for any finite group G and any trivial group, the diagram



induces

$$\mathbb{k} = \mathbf{A}(1) \xrightarrow[\text{unit}]{} \mathbf{A}(G) \xrightarrow[\text{aug}]{} \mathbf{A}(1) = \mathbb{k}$$

where $\operatorname{aug} = \operatorname{res}_{1}^{G}$. Hence $\operatorname{ker} \operatorname{aug} = \mathfrak{m}(G)$ and $A(G)/\mathfrak{m}(G) = \mathbb{k}$. Assumption (B) For any finite group $G, 0 \neq \operatorname{ind}_{1}^{G}(1) \in A(G)$.

Lemma

Suppose that $H \leq K$ and $p \nmid |K : H|$. (i) $\operatorname{ind}_{H}^{K}(1) \in \mathbf{A}(K)^{\times}$. (ii) $\operatorname{res}_{H}^{K} : \mathbf{A}(K) \longrightarrow \mathbf{A}(H)$ is a split monomorphism. In fact $\operatorname{ind}_{H}^{K} \operatorname{res}_{H}^{K}$ is right or left multiplication by a unit in $\mathbf{A}(K)$. If $z \in \ker \operatorname{res}_{H}^{K}$, then by the Frobenius axiom, for $x \in \mathbf{A}(H)$, $z \operatorname{ind}_{H}^{K}(x) = 0 = \operatorname{ind}_{H}^{K}(x)z$.

Similar remarks apply to the restriction α^* associated to an arbitrary group homomorphism α , and to α_* if it is defined.

Lemma

Let G be a finite group. (i) $0 \neq \operatorname{ind}_{1}^{G}(1) \in \operatorname{soc} \boldsymbol{A}(G)$. (ii) If $p \nmid |G|$, then $\operatorname{ind}_{1}^{G}(1) \in \boldsymbol{A}(G)^{\times}$.

The first Lemma and Assumption (A) give

Proposition

Suppose that G is a group for which $p \nmid |G|$. Then $A(G) = \Bbbk$.

The next result is crucial in identifying when a Green functor can be extended to a globally defined Green functor.

Lemma

(a) (p-nilpotent groups) Suppose that P is a p-group, and that N is a group for which $p \nmid |N|$ and P acts on N by automorphisms, so the semi-direct product $PN = P \ltimes N$ is defined. Then the inclusion $P \longrightarrow PN$ induces an isomorphism

$$\operatorname{res}_P^{PN} \colon \boldsymbol{A}(PN) \xrightarrow{\cong} \boldsymbol{A}(P).$$

(b) Suppose that $K \triangleleft G$ where $p \nmid |K|$. Then the quotient epimorphism $\pi: G \longrightarrow G/K$ induces an isomorphism

$$\pi^*: \mathbf{A}(G/K) \xrightarrow{\cong} \mathbf{A}(G).$$

Part (b) allows us to define $\alpha_* = (\alpha^*)^{-1}$ for such a homomorphism $\alpha: G \longrightarrow G/K$, and also for any homomorphism $\beta: G \longrightarrow H$ with $p \nmid |\ker \beta|$.

Theorem

There is a unique extension of **A** to a globally defined local Artinian Green functor for the pair $\mathcal{X}, \mathcal{Y}'$, where \mathcal{Y}' consists of all finite groups of order not divisible by p.

When $N \triangleleft G$, for each $g \in G, c_g$ restricts to an automorphism of A(N), and if $g \in N$ this is the identity. Hence G/N acts on A(N) and we have the invariants $A(N)^{GN}$ and coinvariants $A(N)_{GN}$.

Proposition

Suppose that G has the unique normal p-Sylow subgroup $P \triangleleft G$. Then

$$\boldsymbol{A}(G) = \boldsymbol{A}(P)^{G/P} = \boldsymbol{A}(P)_{G/P}.$$

Furthermore,

$$\mathsf{res}^G_P(\mathsf{ind}^G_1(1)) = |G:P| \mathsf{ind}^P_1(1) \neq 0.$$

Now assume that our local Artinian Green functor \boldsymbol{A} satisfies assumptions (A) and (B). Additionally we assume

Assumption (Frob)

For each finite group G, A(G) is Frobenius, *i.e.*, it is a finite dimensional k-algebra and there is an isomorphism of left A(G)-modules

$$\mathbf{A}(G) \xrightarrow{\cong} \mathbf{A}(G)^* = \operatorname{Hom}_{\Bbbk}(\mathbf{A}(G), \Bbbk).$$

A choice of such an isomorphism determines a Frobenius form $\lambda \in \mathcal{A}(G)^*$ which is the element corresponding to 1. We then refer to the pair $(\mathcal{A}(G), \lambda)$ as a Frobenius algebra (structure) on $\mathcal{A}(G)$. Such a Frobenius form is characterized by the requirement that ker λ contains no non-trivial left (or equivalently right) ideals, which also implies the Gorenstein condition dim_k soc $\mathcal{A}(G) = 1$. In our situation, such a linear form λ is Frobenius if and only if $\lambda(\operatorname{ind}_{1}^{G}(1)) \neq 0$. For a group homomorphism $\alpha \colon G \longrightarrow H$, $\alpha^{*} \colon A(H) \longrightarrow A(G)$ need not send soc A(H) into soc A(G), nor need it be non-zero on it. However, if $\alpha_{*} \colon A(G) \longrightarrow A(H)$ is defined then it restricts to an isomorphism

$$\alpha_*: \operatorname{soc} \mathbf{A}(G) \xrightarrow{\cong} \operatorname{soc} \mathbf{A}(H),$$

since $\alpha_*(\operatorname{ind}_1^G(1)) = \operatorname{ind}_1^H(1) \neq 0$.

Lemma

Suppose that λ is a Frobenius form for A(H). Then $\alpha^* \lambda = \lambda \circ \alpha_*$ is a Frobenius form for A(G).

In this situation, there is a version of Frobenius reciprocity for A(H) and A(G) with suitable compatible inner products.

We continue to make the above assumptions on our Green functor \boldsymbol{A} . In addition we require that it takes values in *commutative* \Bbbk -algebras.

A commutative \Bbbk -algebra valued Green functor A is a Künneth functor if it takes products of groups to pushouts of commutative \Bbbk -algebras, *i.e.*, it satisfies the strict Künneth formula

$$A(G \times H) = A(G) \otimes_{\Bbbk} A(H) = A(G) \otimes A(H)$$

for every pair of finite groups G, H. We will impose another condition.

Assumption (**KF**)

The Green functor **A** is a Künneth functor.

When H = G, the diagonal homomorphism $\Delta: G \longrightarrow G \times G$ induces the product on A(G). If G is an abelian group, the multiplication $G \times G \longrightarrow G$ is a group homomorphism which induces an algebra homomorphism

$$A(G) \longrightarrow A(G \times G) = A(G) \otimes A(G)$$

which is cocommutative, coassociative and counital, with antipode induced by the inverse map $G \longrightarrow G$.

Hence A(G) is naturally a bicommutative Hopf algebra, and by the Larson-Sweedler theorem, it is a Frobenius algebra.

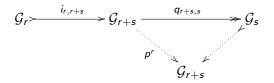
A finite dimensional commutative \Bbbk -Hopf algebra H represents a group scheme Spec(H), i.e., a group valued functor on the category of commutative \Bbbk -algebras \mathscr{CA}_{\Bbbk} ,

$$\operatorname{Spec}(H)(A) = \mathscr{CA}_{\Bbbk}(H, A).$$

Denoting the coproduct by $\psi \colon H \longrightarrow H \otimes H$ and the product on A by $\varphi \colon A \otimes A \longrightarrow A$, we define the group structure by

$$f * g = \varphi(f \otimes g)\psi$$

where $f, g \in \mathscr{CA}_{\Bbbk}(H, A)$. The unit is given by the counit homomorphism $\varepsilon \in \mathscr{CA}_{\Bbbk}(H, \Bbbk)$ and the inverse is given by an algebra automorphism $\chi \in \mathscr{CA}_{\Bbbk}(H, H)$. A sequence of finite abelian group schemes \mathcal{G}_r $(r \ge 1)$ where \mathcal{G}_r has order p^{rh} for some natural number $h \ge 1$ over the field k forms a *p*-divisible group or Barsotti-Tate group of height *h* if there are exact sequences of group schemes fitting into commutative diagrams



for all $r, s \ge 1$, where $\mathcal{G}_r \xrightarrow{i_{r,r+s}} \mathcal{G}_{r+s}$ is a kernel for multiplication by p^r on \mathcal{G}_{r+s} . These are required to be compatible as r, s vary. If $\mathcal{G}_r = \text{Spec}(H_r)$ for some cocommutative Hopf algebra H_r , then $\dim_{\mathbb{K}} H_r = p^{rh}$ and there are morphisms of Hopf algebras

$$H_r \longleftarrow H_{r+s} \longleftarrow H_s$$

inducing the diagram of group schemes

$$\mathcal{G}_r \xrightarrow{i_{r,r+s}} \mathcal{G}_{r+s} \xrightarrow{q_{r+s,s}} \mathcal{G}_s.$$

We are interested in the case where each G_r is connected, which follows from the requirement that each H_r is a local k-algebra. Then there are Borel algebra decompositions:

Proposition

For each $r \ge 1$, there is an isomorphism of \Bbbk -algebras of the form

$$H_r \cong \mathbb{k}[x_1, x_2, \dots, x_{\ell}]/(x_1^{q_1}, x_2^{q_2}, \dots, x_{\ell}^{q_{\ell}}),$$

where $q_i = p^{d_i}$ for some $d_i \ge 1$ and $d_1 + d_2 + \dots + d_{\ell} = rh$. Notice that soc $H_r = \mathbb{k}\{x_1^{q_1-1}x_2^{q_2-1}\cdots x_{\ell}^{q_{\ell}-1}\}.$ We now introduce another assumption.

Assumption (D)

The Hopf algebras $\mathbf{A}(C_{p^r})$ with restriction and inflation homomorphisms $\operatorname{res}_{C_{p^r}}^{C_{p^{r+s}}}$ and $\operatorname{res}_{C_{p^{r+s}}}^{C_{p^s}}$ (induced by the canonical quotient homomorphism $C_{p^{r+s}} \longrightarrow C_{p^s}$) give rise to a *p*-divisible group with $\mathcal{G}_r = \operatorname{Spec} \mathbf{A}(C_{p^r})$.

We can deduce a basic non-triviality result.

Proposition

Suppose that $\pi: G \longrightarrow C_{p^s}$ be an epimorphism for $s \ge 1$. Then the induced homomorphism $\pi^*: \mathbf{A}(C_{p^s}) \longrightarrow \mathbf{A}(G)$ is a monomorphism.

Corollary

If G is a non-trivial p-group, then $\mathbf{A}(G)$ is non-trivial, $\mathbf{A}(G) \neq \Bbbk$. In particular, $0 \neq \operatorname{ind}_{1}^{G}(1) \notin \Bbbk$.

Using the Mackey formula we obtain

Theorem

Let G be a finite group. Then $\mathbf{A}(G) = \mathbb{k}$ if and only if $p \nmid |G|$.

Are there interesting examples of such Green functors other than ones related to Morava K-theories?

Atiyah-Segal: When n = 1, a completion of the usual complex representation ring R(G) reduced modulo p gives something close to $K_1^0(BG)$.

Hopkins-Kuhn-Ravenel: When n > 1, there is a theory of generalised character rings which produces a model for $K_n^0(BG)$ when $K_n^1(BG) = 0$.

Character rings

Let R(G) be the character or representation ring of G, and let $\overline{R}(G) = \mathbb{F}_p \otimes R(G)$. Then for a p-Sylow subgroup $P \leq G$, restriction induces a homomorphism $\rho \colon \overline{R}(G) \longrightarrow \overline{R}(P)$ and $\overline{R}(P)$ is known to be local, hence $\overline{R}(G)/\ker \rho$ is local. Here the Frobenius structure can be traced back to one on $\overline{R}(G)$ coming from a linear form on characters,

 $\chi \mapsto$ multiplicty of trivial character in χ .

Example: $G = S_3$, p = 2. Here

 $\overline{\mathrm{R}}(S_3) = \mathbb{F}_2[a, b]/(a^2, b^2 + b + a), \quad \overline{\mathrm{R}}(\langle (12) \rangle) = \mathbb{F}_2[c]/(c^2),$

and the restriction is given by $a \mapsto c$, $b \mapsto c$, so

$$\overline{\mathrm{R}}(S_3)/\ker\rho\cong\mathbb{F}_2[a](a^2).$$

Schuster: For any n,

$$K_n^0(BQ_8) = \mathbb{F}_2[y_1, y_2, c_2]/(\text{relations}),$$

where the ideal of relations is generated by

$$y_1^{2^n}, y_2^{2^n}, c_2^{2^n} + y_1^2 + y_1y_2 + y_2^2$$

and

$$\sum_{k=1}^{n} y_1^{2^n - 2^k + 1} c_2^{2^{k-1}}, \quad \sum_{k=1}^{n} y_2^{2^n - 2^k + 1} c_2^{2^{k-1}}.$$

Here the socle is generated by $c_2^{2^{n+1}-1} = y^{2^n-1}y_2^{2^n-1}$.