Frobenius Green functors

Andrew Baker, University of Glasgow/MSRI

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Topological Motivation: Morava K-theory and finite groups

For each prime p and each natural number n there is a 2-periodic multiplicative cohomology theory $K_n^*(-)$ with coefficient ring

$$K_n^*(\mathrm{point}) = K_n^* = \mathbb{F}[u, u^{-1}],$$

where $u \in K_n^{-2}$ and $\mathbb{F}$ is either $\mathbb{F}_{p^n}$ or $\overline{\mathbb{F}}_p$. Here K_n^* is a graded field in an obvious sense; computations are often simplified with the aid of a strict Künneth formula for products:

$$K_n^*(X \times Y) \cong K_n^*(X) \otimes_{K_n^*} K_n^*(Y).$$

For each finite group G, $K_n^*(BG)$ is finite dimensional over K_n^* and $K_n^*(BG)$ is self-dual as a $K_n^*(BG)$-module and so is self-injective. Since $K_n^*(BG)$ is also a local K_n^*-algebra, this means that $K_n^*(BG)$ is a local Frobenius algebra. Another important observation is that $\mathrm{Tr}_1^G(1) \neq 0$, and it follows that $\mathrm{Tr}_1^G(1)$ is a basis element for the socle of $K_n^*(BG)$. $K_n^*(B-)$ is a *Green functor* on subgroups of a fixed finite group G.

p-primary field-based cohomology theories

$$
\begin{tikzcd}
 & & & BP^*(-) \arrow[dlll] \arrow[ddlll] \arrow[ddll] \arrow[ddl] & \\
H^*(-;\mathbb{F}_p) & & & & \\
H^*(-;\mathbb{Z}_{(p)}) \arrow[u] \arrow[d] & BP\langle 1\rangle^*(-) \arrow[l] \arrow[d] & BP\langle 2\rangle^*(-) \arrow[l] \arrow[d] & \cdots \arrow[l] & \cdots \\
H^*(-;\mathbb{Q}) \arrow[d, equal] & K_1^*(-) & K_2^*(-) & & \\
K_0^*(-) & & & &
\end{tikzcd}
$$

Mackey functors and Green functors for finite groups

Let G be a finite group. Consider the category $\mathbf{SubGp}_G$ whose objects are the subgroups of G and the morphism set $\mathbf{SubGp}_G(H, K)$ consists of all conjugation maps

$$c_g = c_g^{H,K} \colon H \longrightarrow K; \quad x \mapsto gxg^{-1}$$

for $g \in G$ where $gHg^{-1} \leqslant K$. Here $c_g^{H,K}$ can be factorized as

$$\begin{tikzcd}
 & H \arrow[dl, "c_g^{H,gHg^{-1}}"'] \arrow[dd, "c_g^{H,K}"'] \arrow[dr, hook] & \\
gHg^{-1} \arrow[dr, hook] & & g^{-1}Kg \arrow[dl, "c_g^{g^{-1}Kg,K}"] \\
 & K &
\end{tikzcd}$$

where $c_g^{H,gHg^{-1}}$ and $c_g^{g^{-1}Kg,K}$ are isomorphisms and $\hookrightarrow$ denotes the inclusion of a subgroup.

Let $\Bbbk$ be a commutative ring. Roughly speaking, a *Mackey functor* $\boldsymbol{M}$ for G consists of a contravariant functor $\boldsymbol{M}^*$ and a covariant functor $\boldsymbol{M}_*$ on $\mathbf{SubGp}_G$ taking values in (left) $\Bbbk$-modules, where

$$\boldsymbol{M}(H) = \boldsymbol{M}^*(H) = \boldsymbol{M}_*(H),$$

and there are various axioms describing the interaction between these. This amounts to the following.

For $K \leqslant H \leqslant G$ and $g \in G$ there are morphisms

$$\mathrm{res}^H_K \colon \boldsymbol{M}(H) \longrightarrow \boldsymbol{M}(K), \qquad \mathrm{ind}^H_K \colon \boldsymbol{M}(K) \longrightarrow \boldsymbol{M}(H),$$
$$c_g \colon \boldsymbol{M}(H) \longrightarrow \boldsymbol{M}(gHg^{-1}).$$

These satisfy the following conditions.

(**MF1**) For $H \leqslant G$ and $h \in H$,

$$\mathrm{res}^H_H = \mathrm{ind}^H_H = c_h = \mathrm{id}\colon \boldsymbol{M}(H) \longrightarrow \boldsymbol{M}(H).$$

(**MF2**) For $L \leqslant K \leqslant H \leqslant G$,

$$\mathrm{res}^K_L \, \mathrm{res}^H_K = \mathrm{res}^H_L, \quad \mathrm{ind}^H_K \, \mathrm{ind}^K_L = \mathrm{ind}^H_L \, .$$

(**MF3**) For $g_1, g_2 \in G$ and $H \leqslant G$,

$$c_{g_1} c_{g_2} = c_{g_1 g_2}\colon \boldsymbol{M}(H) \longrightarrow \boldsymbol{M}(g_1 g_2 H g_2^{-1} g_1^{-1}).$$

(**MF4**) For $K \leqslant H \leqslant G$ and $g \in G$,

$$\mathrm{res}^{gHg^{-1}}_{gKg^{-1}} \, c_g = c_g \, \mathrm{res}^H_K, \quad \mathrm{ind}^{gHg^{-1}}_{gKg^{-1}} \, c_g = c_g \, \mathrm{ind}^H_K \, .$$

(**MF5**) (*Mackey double coset/decomposition formula*) For $H \leqslant G$ and $K \leqslant H \geqslant L$,

$$\mathrm{res}^H_L \, \mathrm{ind}^H_K = \sum_{g\colon L\backslash G/K} \mathrm{ind}^L_{L\cap gKg^{-1}} \, c_g \, \mathrm{res}^K_{g^{-1}Lg\cap K},$$

where the sum is over a complete set of representatives for the set of double cosets $L\backslash G/K$.

Such a Mackey functor $\boldsymbol{A}$ is a *Green functor* if furthermore $\boldsymbol{A}(H)$ is a $\Bbbk$-algebra for $H \leqslant G$, and the following conditions are satisfied.
(**GF1**) For $K \leqslant H \leqslant G$ and $g \in G$, res^H_K and c_g are $\Bbbk$-algebra homomorphisms.
(**GF2**) (*Frobenius axiom*) For $K \leqslant H \leqslant G$, $x \in \boldsymbol{A}(K)$ and $y \in \boldsymbol{A}(H)$,

$$\mathrm{ind}^H_K(x\,\mathrm{res}^H_K(y)) = \mathrm{ind}^H_K(x)y, \quad \mathrm{ind}^H_K(\mathrm{res}^H_K(y)x) = y\,\mathrm{ind}^H_K(x).$$

Notice that $\boldsymbol{A}(K)$ becomes both a left and a right $\boldsymbol{A}(H)$-module via $\mathrm{res}^H_K(y)$, so for $x \in \boldsymbol{A}(K)$ and $y \in \boldsymbol{A}(H)$,

$$y \cdot x = \mathrm{res}^H_K(y)x, \quad x \cdot y = x\,\mathrm{res}^H_K(y).$$

The Frobenius axiom asserts that $\mathrm{ind}^H_K \colon \boldsymbol{A}(K) \longrightarrow \boldsymbol{A}(H)$ is both a left and a right $\boldsymbol{A}(H)$-module homomorphism.

It is often useful to consider extensions of a Mackey functor or Green functor to a larger category.

We say that K is a *section* of a group G if there is a subgroup $H \leqslant G$ and an epimorphism $H \longrightarrow K$.

Suppose that $\mathcal{X}, \mathcal{Y}$ are two collections of finite groups where $\mathcal{X}$ satisfies the following conditions.

- If $G \in \mathcal{X}$ and if K is a section of G, then $K \in \mathcal{X}$.
- Let $G', G'' \in \mathcal{X}$. If

$$1 \to G' \longrightarrow G \longrightarrow G'' \to 1,$$

  is a short exact sequence, then $G \in \mathcal{X}$.

A *globally defined Mackey functor* with respect to $\mathcal{X}, \mathcal{Y}$ on finite groups and taking values in $\Bbbk$-modules, is an assignment of an $\Bbbk$-module $\boldsymbol{M}(G)$ to each finite group G, and for each homomorphism $\alpha\colon G \longrightarrow H$ with $\ker \alpha \in \mathcal{X}$ a homomorphism $\alpha^*\colon \boldsymbol{M}(H) \longrightarrow \boldsymbol{M}(G)$, and for each homomorphism $\beta\colon K \longrightarrow L$ with $\ker \alpha \in \mathcal{Y}$ a homomorphism $\beta_*\colon \boldsymbol{M}(K) \longrightarrow \boldsymbol{M}(L)$, satisfying the following conditions.

(**GD1**) When these are defined, $(\alpha_1\alpha_2)^* = \alpha_2^*\alpha_1^*$ and $(\beta_1\beta_2)_* = (\beta_1)_*(\beta_2)_*$.
(**GD2**) If $\gamma\colon G \longrightarrow G$ is an inner automorphism, then $\alpha^* = \mathrm{id} = \alpha_*$.
(**GD3**) For a commutative diagram of finite groups

$$\begin{tikzcd}
\alpha^{-1}K \arrow[r, tail, "\gamma"] \arrow[d, two heads, "\beta"'] & G \arrow[d, two heads, "\alpha"] \\
K \arrow[r, tail, "\delta"] & H
\end{tikzcd}$$

we have $\delta^*\alpha_* = \beta_*\gamma^*$ whenever $\ker\alpha \in \mathcal{Y}$, and $\alpha^*\delta_* = \gamma_*\beta^*$ whenever $\ker\alpha \in \mathcal{X}$.

(**GD4**) For a commutative diagram of epimorphisms of finite groups

$$\begin{tikzcd} G \arrow[r, two heads, "\beta"] \arrow[d, two heads, "\alpha"'] & K \arrow[d, two heads, "\gamma"] \\ H \arrow[r, two heads, "\delta"] & G/\ker\alpha\ker\beta \end{tikzcd}$$

with $\ker\alpha \in \mathcal{Y}$ and $\ker\beta \in \mathcal{X}$, we have $\alpha_*\beta^* = \delta^*\gamma_*$.

(**GD5**) (Mackey formula) For subgroups $H \leqslant G \geqslant K$ with inclusion maps $\iota_H^G\colon H \longrightarrow G$, etc, and conjugation maps $c_g = g(-)g^{-1}\colon g^{-1}Kg \cap H \longrightarrow K \cap gHg^{-1}$,

$$(\iota_K^G)^*(\iota_H^G)_* = \sum_{g\colon K\backslash G/H} (\iota_{K\cap gHg^{-1}}^K)_* c_g (\iota_{g^{-1}Kg\cap H}^H)^*.$$

Such a globally defined Mackey functor $\boldsymbol{A}$ is a *globally defined Green functor* if

(**GD6**) Whenever $\alpha \colon G \longrightarrow H$ is a homomorphism for which α^* is defined, then $\alpha^* \colon \boldsymbol{A}(H) \longrightarrow \boldsymbol{A}(G)$ is an $\Bbbk$-algebra homomorphism.

(**GD7**) (Frobenius axiom) Suppose that $\beta \colon K \longrightarrow L$ is a homomorphism for which β^* and β_* are both defined. Note that β^* induces left and right $\boldsymbol{A}(H)$-module structures on $\boldsymbol{A}(G)$ (these coincide when $\boldsymbol{A}(G)$ is commutative). Then $\beta_* \colon \boldsymbol{A}(G) \longrightarrow \boldsymbol{A}(H)$ is a homomorphism of left and right $\boldsymbol{A}(H)$-modules.

For our purposes we will take $\mathcal{X}$ to consist of all finite groups, and $\mathcal{Y}$ to consist of either trivial groups or all groups of order not divisible by a positive prime $p = \operatorname{char} \Bbbk$.

Local Artinian Green functors

Now we assume that $\boldsymbol{A}$ is a globally defined Green functor taking values in the category of local Artinian $\Bbbk$-algebras, for $\Bbbk$ a field with $\operatorname{char} \Bbbk = p > 0$. The unique maximal left ideal, $\mathfrak{m}(G) \lhd \boldsymbol{A}(G)$, agrees with the Jacobson radical, $\mathfrak{m}(G) = \operatorname{rad} \boldsymbol{A}(G)$, which is a two-sided ideal.

We take $\mathcal{X}$ to consist of all finite groups and $\mathcal{Y}$ to consist of the trivial groups. The restriction/inflation homomorphism associated to a homomorphism $\alpha\colon G \longrightarrow H$ will be denoted $\alpha^*\colon \boldsymbol{A}(H) \longrightarrow \boldsymbol{A}(G)$ and the induction homomorphism associated to a monomorphism $\beta\colon K \longrightarrow L$ by $\beta_*\colon \boldsymbol{A}(K) \longrightarrow \boldsymbol{A}(L)$. When β is the inclusion of a subgroup we will also write $\beta^* = \operatorname{res}^L_K$ and $\beta_* = \operatorname{ind}^L_K$.

Here α^* is local algebra homomorphism, *i.e.*, $\alpha^*\mathfrak{m}(H) \subseteq \mathfrak{m}(G)$, while β_* is a left or right $\boldsymbol{A}(L)$-module homomorphism with respect to the $\boldsymbol{A}(L)$-module structure induced on $\boldsymbol{A}(K)$ by β^*.

We make some further assumptions.

Assumption (**A**)

For any trivial group 1, $\boldsymbol{A}(1) = \Bbbk$.

Notice that for any finite group G and any trivial group, the diagram

$$\begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left, "="] & G \arrow[r] & 1 \end{tikzcd}$$

induces

$$\begin{tikzcd} \Bbbk = \boldsymbol{A}(1) \arrow[r, "\mathrm{unit}"'] \arrow[rr, bend left, "="] & \boldsymbol{A}(G) \arrow[r, "\mathrm{aug}"'] & \boldsymbol{A}(1) = \Bbbk \end{tikzcd}$$

where $\mathrm{aug} = \mathrm{res}^G_1$. Hence $\ker \mathrm{aug} = \mathfrak{m}(G)$ and $\boldsymbol{A}(G)/\mathfrak{m}(G) = \Bbbk$.

Assumption (**B**)

For any finite group G, $0 \neq \mathrm{ind}^G_1(1) \in \boldsymbol{A}(G)$.

Lemma

Suppose that $H \leqslant K$ *and* $p \nmid |K : H|$.
(i) $\mathrm{ind}_H^K(1) \in \boldsymbol{A}(K)^\times$.
(ii) $\mathrm{res}_H^K \colon \boldsymbol{A}(K) \longrightarrow \boldsymbol{A}(H)$ *is a split monomorphism. In fact* $\mathrm{ind}_H^K \mathrm{res}_H^K$ *is right or left multiplication by a unit in* $\boldsymbol{A}(K)$.

If $z \in \ker \mathrm{res}_H^K$, then by the Frobenius axiom, for $x \in \boldsymbol{A}(H)$,

$$z \, \mathrm{ind}_H^K(x) = 0 = \mathrm{ind}_H^K(x) z.$$

Similar remarks apply to the restriction α^* associated to an arbitrary group homomorphism α, and to α_* if it is defined.

Lemma

Let G *be a finite group.*
(i) $0 \neq \mathrm{ind}_1^G(1) \in \mathrm{soc}\, \boldsymbol{A}(G)$.
(ii) If $p \nmid |G|$, *then* $\mathrm{ind}_1^G(1) \in \boldsymbol{A}(G)^\times$.

The first Lemma and Assumption (A) give

Proposition

Suppose that G *is a group for which* $p \nmid |G|$. *Then* $\boldsymbol{A}(G) = \Bbbk$.

The next result is crucial in identifying when a Green functor can be extended to a globally defined Green functor.

Lemma

(a) (p-nilpotent groups) *Suppose that P is a p-group, and that N is a group for which $p \nmid |N|$ and P acts on N by automorphisms, so the semi-direct product $PN = P \ltimes N$ is defined. Then the inclusion $P \longrightarrow PN$ induces an isomorphism*

$$\mathrm{res}^{PN}_{P} \colon \boldsymbol{A}(PN) \xrightarrow{\cong} \boldsymbol{A}(P).$$

(b) Suppose that $K \triangleleft G$ where $p \nmid |K|$. Then the quotient epimorphism $\pi \colon G \longrightarrow G/K$ induces an isomorphism

$$\pi^* \colon \boldsymbol{A}(G/K) \xrightarrow{\cong} \boldsymbol{A}(G).$$

Part (b) allows us to define $\alpha_* = (\alpha^*)^{-1}$ for such a homomorphism $\alpha\colon G \longrightarrow G/K$, and also for any homomorphism $\beta\colon G \longrightarrow H$ with $p \nmid |\ker \beta|$.

Theorem

There is a unique extension of $\boldsymbol{A}$ to a globally defined local Artinian Green functor for the pair $\mathcal{X}, \mathcal{Y}'$, where $\mathcal{Y}'$ consists of all finite groups of order not divisible by p.

When $N \lhd G$, for each $g \in G$,c_g restricts to an automorphism of $\boldsymbol{A}(N)$, and if $g \in N$ this is the identity. Hence G/N acts on $\boldsymbol{A}(N)$ and we have the invariants $\boldsymbol{A}(N)^{GN}$ and coinvariants $\boldsymbol{A}(N)_{GN}$.

Proposition

Suppose that G has the unique normal p-Sylow subgroup $P \lhd G$. Then

$$\boldsymbol{A}(G) = \boldsymbol{A}(P)^{G/P} = \boldsymbol{A}(P)_{G/P}.$$

Furthermore,

$$\mathrm{res}_P^G(\mathrm{ind}_1^G(1)) = |G : P| \,\mathrm{ind}_1^P(1) \neq 0.$$

Frobenius Green functors

Now assume that our local Artinian Green functor $\boldsymbol{A}$ satisfies assumptions (A) and (B). Additionally we assume

Assumption (**Frob**)

For each finite group G, $\boldsymbol{A}(G)$ is Frobenius, *i.e.*, it is a finite dimensional $\Bbbk$-algebra and there is an isomorphism of left $\boldsymbol{A}(G)$-modules

$$\boldsymbol{A}(G) \xrightarrow{\cong} \boldsymbol{A}(G)^* = \mathrm{Hom}_{\Bbbk}(\boldsymbol{A}(G), \Bbbk).$$

A choice of such an isomorphism determines a Frobenius form $\lambda \in \boldsymbol{A}(G)^*$ which is the element corresponding to 1. We then refer to the pair $(\boldsymbol{A}(G), \lambda)$ as a Frobenius algebra (structure) on $\boldsymbol{A}(G)$. Such a Frobenius form is characterized by the requirement that $\ker \lambda$ contains no non-trivial left (or equivalently right) ideals, which also implies the Gorenstein condition $\dim_{\Bbbk} \operatorname{soc} \boldsymbol{A}(G) = 1$.

In our situation, such a linear form λ is Frobenius if and only if $\lambda(\mathrm{ind}_1^G(1)) \neq 0$.

For a group homomorphism $\alpha\colon G \longrightarrow H$, $\alpha^*\colon \boldsymbol{A}(H) \longrightarrow \boldsymbol{A}(G)$ need not send $\operatorname{soc} \boldsymbol{A}(H)$ into $\operatorname{soc} \boldsymbol{A}(G)$, nor need it be non-zero on it. However, if $\alpha_*\colon \boldsymbol{A}(G) \longrightarrow \boldsymbol{A}(H)$ is defined then it restricts to an isomorphism

$$\alpha_*\colon \operatorname{soc} \boldsymbol{A}(G) \xrightarrow{\cong} \operatorname{soc} \boldsymbol{A}(H),$$

since $\alpha_*(\mathrm{ind}_1^G(1)) = \mathrm{ind}_1^H(1) \neq 0$.

Lemma

Suppose that λ is a Frobenius form for $\boldsymbol{A}(H)$. Then $\alpha^*\lambda = \lambda \circ \alpha_*$ is a Frobenius form for $\boldsymbol{A}(G)$.

In this situation, there is a version of Frobenius reciprocity for $\boldsymbol{A}(H)$ and $\boldsymbol{A}(G)$ with suitable compatible inner products.

We continue to make the above assumptions on our Green functor $\boldsymbol{A}$. In addition we require that it takes values in *commutative* $\mathbb{k}$-algebras.

A commutative $\mathbb{k}$-algebra valued Green functor $\boldsymbol{A}$ is a *Künneth functor* if it takes products of groups to pushouts of commutative $\mathbb{k}$-algebras, *i.e.*, it satisfies the strict Künneth formula

$$\boldsymbol{A}(G \times H) = \boldsymbol{A}(G) \otimes_{\mathbb{k}} \boldsymbol{A}(H) = \boldsymbol{A}(G) \otimes \boldsymbol{A}(H)$$

for every pair of finite groups G, H.

We will impose another condition.

Assumption (KF)

The Green functor $\boldsymbol{A}$ is a Künneth functor.

When $H = G$, the diagonal homomorphism $\Delta\colon G \longrightarrow G \times G$ induces the product on $\boldsymbol{A}(G)$. If G is an abelian group, the multiplication $G \times G \longrightarrow G$ is a group homomorphism which induces an algebra homomorphism

$$\boldsymbol{A}(G) \longrightarrow \boldsymbol{A}(G \times G) = \boldsymbol{A}(G) \otimes \boldsymbol{A}(G)$$

which is cocommutative, coassociative and counital, with antipode induced by the inverse map $G \longrightarrow G$.
Hence $\boldsymbol{A}(G)$ is naturally a bicommutative Hopf algebra, and by the Larson-Sweedler theorem, it is a Frobenius algebra.

A finite dimensional commutative $\Bbbk$-Hopf algebra H represents a group scheme $\mathrm{Spec}(H)$, i.e., a group valued functor on the category of commutative $\Bbbk$-algebras $\mathscr{C}\mathscr{A}_{\Bbbk}$,

$$\mathrm{Spec}(H)(A) = \mathscr{C}\mathscr{A}_{\Bbbk}(H, A).$$

Denoting the coproduct by $\psi\colon H \longrightarrow H \otimes H$ and the product on A by $\varphi\colon A \otimes A \longrightarrow A$, we define the group structure by

$$f * g = \varphi(f \otimes g)\psi$$

where $f, g \in \mathscr{C}\mathscr{A}_{\Bbbk}(H, A)$. The unit is given by the counit homomorphism $\varepsilon \in \mathscr{C}\mathscr{A}_{\Bbbk}(H, \Bbbk)$ and the inverse is given by an algebra automorphism $\chi \in \mathscr{C}\mathscr{A}_{\Bbbk}(H, H)$.

A sequence of finite abelian group schemes $\mathcal{G}_r$ $(r \geqslant 1)$ where $\mathcal{G}_r$ has order p^{rh} for some natural number $h \geqslant 1$ over the field $\Bbbk$ forms a *p-divisible group* or *Barsotti-Tate group* of height h if there are exact sequences of group schemes fitting into commutative diagrams

$$\begin{tikzcd}
\mathcal{G}_r \arrow[r, tail, "i_{r,r+s}"] & \mathcal{G}_{r+s} \arrow[rr, two heads, "q_{r+s,s}"] \arrow[dr, dotted, "p^r"'] & & \mathcal{G}_s \arrow[dl, dotted] \\
& & \mathcal{G}_{r+s} &
\end{tikzcd}$$

for all $r, s \geqslant 1$, where $\mathcal{G}_r \xrightarrow{i_{r,r+s}} \mathcal{G}_{r+s}$ is a kernel for multiplication by p^r on $\mathcal{G}_{r+s}$. These are required to be compatible as r, s vary.

If $\mathcal{G}_r = \mathrm{Spec}(H_r)$ for some cocommutative Hopf algebra H_r, then $\dim_{\Bbbk} H_r = p^{rh}$ and there are morphisms of Hopf algebras

$$H_r \twoheadleftarrow H_{r+s} \leftarrowtail H_s$$

inducing the diagram of group schemes

$$\mathcal{G}_r \xrightarrow{i_{r,r+s}} \mathcal{G}_{r+s} \xrightarrow{q_{r+s,s}} \mathcal{G}_s.$$

We are interested in the case where each $\mathcal{G}_r$ is connected, which follows from the requirement that each H_r is a local $\Bbbk$-algebra. Then there are Borel algebra decompositions:

Proposition

For each $r \geqslant 1$, there is an isomorphism of $\Bbbk$-algebras of the form

$$H_r \cong \Bbbk[x_1, x_2, \ldots, x_\ell]/(x_1^{q_1}, x_2^{q_2}, \ldots, x_\ell^{q_\ell}),$$

where $q_i = p^{d_i}$ for some $d_i \geqslant 1$ and $d_1 + d_2 + \cdots + d_\ell = rh$.

Notice that $\operatorname{soc} H_r = \Bbbk\{x_1^{q_1-1} x_2^{q_2-1} \cdots x_\ell^{q_\ell-1}\}$.

We now introduce another assumption.

Assumption (**D**)

The Hopf algebras $\boldsymbol{A}(C_{p^r})$ with restriction and inflation homomorphisms $\mathrm{res}^{C_{p^{r+s}}}_{C_{p^r}}$ and $\mathrm{res}^{C_{p^s}}_{C_{p^{r+s}}}$ (induced by the canonical quotient homomorphism $C_{p^{r+s}} \longrightarrow C_{p^s}$) give rise to a p-divisible group with $\mathcal{G}_r = \operatorname{Spec} \boldsymbol{A}(C_{p^r})$.

We can deduce a basic non-triviality result.

Proposition

Suppose that $\pi\colon G \longrightarrow C_{p^s}$ be an epimorphism for $s \geqslant 1$. Then the induced homomorphism $\pi^*\colon \boldsymbol{A}(C_{p^s}) \longrightarrow \boldsymbol{A}(G)$ is a monomorphism.

Corollary

If G is a non-trivial p-group, then $\boldsymbol{A}(G)$ is non-trivial, $\boldsymbol{A}(G) \neq \Bbbk$. In particular, $0 \neq \mathrm{ind}_1^G(1) \notin \Bbbk$.

Using the Mackey formula we obtain

Theorem

Let G be a finite group. Then $\boldsymbol{A}(G) = \Bbbk$ if and only if $p \nmid |G|$.

Question

Are there interesting examples of such Green functors other than ones related to Morava K-theories?

Atiyah-Segal: When $n = 1$, a completion of the usual complex representation ring $R(G)$ reduced modulo p gives something close to $K_1^0(BG)$.

Hopkins-Kuhn-Ravenel: When $n > 1$, there is a theory of generalised character rings which produces a model for $K_n^0(BG)$ when $K_n^1(BG) = 0$.

Character rings

Let $\mathrm{R}(G)$ be the character or representation ring of G, and let $\overline{\mathrm{R}}(G) = \mathbb{F}_p \otimes \mathrm{R}(G)$. Then for a p-Sylow subgroup $P \leqslant G$, restriction induces a homomorphism $\rho\colon \overline{\mathrm{R}}(G) \longrightarrow \overline{\mathrm{R}}(P)$ and $\overline{\mathrm{R}}(P)$ is known to be local, hence $\overline{\mathrm{R}}(G)/\ker\rho$ is local. Here the Frobenius structure can be traced back to one on $\overline{\mathrm{R}}(G)$ coming from a linear form on characters,

$$\chi \mapsto \text{multiplicty of trivial character in } \chi.$$

Example: $G = S_3$, $p = 2$. Here

$$\overline{\mathrm{R}}(S_3) = \mathbb{F}_2[a,b]/(a^2, b^2 + b + a), \quad \overline{\mathrm{R}}(\langle(1\,2)\rangle) = \mathbb{F}_2[c]/(c^2),$$

and the restriction is given by $a \mapsto c$, $b \mapsto c$, so

$$\overline{\mathrm{R}}(S_3)/\ker\rho \cong \mathbb{F}_2[a](a^2).$$

Morava K-theory of Q_8

Schuster: For any n,

$$K_n^0(BQ_8) = \mathbb{F}_2[y_1, y_2, c_2]/(\text{relations}),$$

where the ideal of relations is generated by

$$y_1^{2^n},\ y_2^{2^n},\ c_2^{2^n} + y_1^2 + y_1 y_2 + y_2^2,$$

and

$$\sum_{k=1}^{n} y_1^{2^n - 2^k + 1} c_2^{2^{k-1}}, \quad \sum_{k=1}^{n} y_2^{2^n - 2^k + 1} c_2^{2^{k-1}}.$$

Here the socle is generated by $c_2^{2^{n+1}-1} = y^{2^n-1} y_2^{2^n-1}$.